This concludes the proof.

Remark. On sets without interior, a considerably stronger version of the theorem can be derived from it without any extra effort:

If E is an Arakelian set with empty interior, f and ω are continuous functions on E, f is complex-valued, ω is positive (and $\omega(z) \to 0$ as $z \to \infty$ along E, to make things interesting), then there is an entire function h that satisfies

$$|h(z) - f(z)| < \omega(z)$$

for every $z \in E$.

To prove this, apply the theorem twice: There are entire functions g and h_0 so that

$$\operatorname{Re} g < \log \omega \text{ and } |h_0 - f \cdot \exp(-g)| < 1$$

on E. Put $h = h_0 \cdot \exp(g)$.

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Added in proof: Another reference to Arakelian's theorem is Approximation Uniforme Qualitative sur les Ensembles non Bornés, by P. M. Gauthier and W. Hengartner, Presses de l'Université de Montréal, 1962, p. 37.

The Norm of a Linear Functional

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Bounded linear functionals of the type

$$f(x) = \int_a^b g(t)x(t)\,dt \tag{1}$$

frequently occur in elementary functional analysis and its applications, and one needs to have an expression for $\|f\|$, the norm of f. For example, if $x = x(t)$ is a continuous function of period 2π and X is the Banach space of all such functions, with $\|x\| = \max\{|x(t)|: -\pi \leqslant t \leqslant \pi\}$, then the Fourier coefficients of x are, by definition,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\cos kt\,dt, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin kt\,dt, \tag{2}$$

for $k = 0, 1, 2, \dots$. Now if $s_n(x)$ denotes the nth partial sum of the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kw + b_k \sin kw)$$

in the case when $w = 0$, we see from (2) that

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 + 2 \sum_{k=1}^{n} \cos kt\right) x(t)\, dt, \tag{3}$$

which is of the form (1), with x a continuous function on $[-\pi, \pi]$ and with

$$g(t) = \frac{1}{2\pi}\left(1 + 2 \sum_{k=1}^{n} \cos kt\right) = \frac{1}{2\pi} \cdot \frac{\sin(n + 1/2)t}{\sin(t/2)}. \tag{4}$$

It follows from (3) and (4) that

$$|s_n(x)| \leqslant \frac{1}{2\pi}\left(1 + \sum_{k=1}^{n} 2\right) \int_{-\pi}^{\pi} |x(t)|\, dt \leqslant (1 + 2n)\|x\|,$$

which implies that s_n is a bounded (and obviously linear) functional on X. Since, by definition, $\|s_n\| = \sup\{|s_n(x)|: \|x\| \leqslant 1\}$, it follows that $\|s_n\| \leqslant 1 + 2n$.

In order to deduce the very interesting result that there is a continuous function of period 2π whose Fourier series diverges at 0 it is enough to show that $\|s_n\| \to \infty$ as $n \to \infty$. Details may be found in Gål [**1**], where it is shown that

$$\|s_n\| = \int_{-\pi}^{\pi} |g(t)|\, dt, \tag{5}$$

with g given by (4).

Because of the especially simple form of g in (5) the determination of $\|s_n\|$ is not difficult, but in the more general case (1), where x may be any continuous function on $[a, b]$ and g any given continuous function the usual proofs which determine $\|f\|$ are quite technical and reasonably lengthy. See, for example, Kantorovich and Akilov [**2**], p. 104.

In this note we show how to determine $\|f\|$ in (1) by a short elementary method which holds also for any Riemann integrable functions.

Now let us denote by $R[a, b]$ the real linear space of Riemann integrable functions $x = x(t)$, $a \leqslant t \leqslant b$, with a and b finite real numbers. Define the norm of x to be $\|x\| = \sup\{|x(t)|: a \leqslant t \leqslant b\}$. Since every continuous x is in $R[a, b]$ we have that $C[a, b]$ is a linear subspace of $R[a, b]$ and so the theorem below also applies to any continuous functions.

THEOREM 1. *Let g be a fixed element of $R[a, b]$. Then f given by* (1) *above defines a bounded linear functional on $R[a, b]$ and*

$$\|f\| = \sup\{|f(x)|: \|x\| \leqslant 1\} = \int_a^b |g(t)|\, dt. \tag{6}$$

Proof. First, for any $x \in R[a, b]$ with $\|x\| \leqslant 1$ we have

$$|f(x)| \leqslant \int_a^b |g(t)| \cdot |x(t)|\, dt \leqslant \|x\| \int_a^b |g(t)|\, dt,$$

which implies $\|f\| \leqslant \int_a^b |g(t)|\,dt$. The problem is to prove the reverse inequality. To do this, take any natural number n. Then, writing for simplicity, $\int|g| = \int_a^b |g(t)|\,dt$, etc., we have

$$\begin{aligned}\int|g| &= \int|g| \cdot \frac{1+n|g|}{1+n|g|}\\ &= \int \frac{|g|}{1+n|g|} + \int g \cdot \frac{ng}{1+n|g|} = \int \frac{|g|}{1+n|g|} + f\left(\frac{ng}{1+n|g|}\right)\\ &\leqslant \int \frac{dt}{n} + \|f\| \left\| \frac{ng}{1+n|g|} \right\|,\end{aligned}$$

using the facts that $|g(t)|/(1+n|g(t)|) < 1/n$ and that $|f(x)| \leqslant \|f\|\,\|x\|$ for all $x \in R[a, b]$. Also, since $\|ng/(1+n|g|)\| \leqslant 1$, we see that

$$\int_a^b |g(t)|\,dt \leqslant (b-a)/n + \|f\|, \tag{7}$$

and our result follows by letting $n \to \infty$ in (7).

An extension of Theorem 1 is obtained by considering the space $R_B[a, \infty)$ of Riemann integrable bounded functions x on $[a, \infty)$ with $\|x\| = \sup\{|x(t)|\colon a \leqslant t < \infty\}$, and a fixed $g \in R[a, y]$ for each $y > a$ which is such that

$$\lim_{y \to \infty} \int_a^y |g(t)|\,dt = \int_a^\infty |g(t)|\,dt < \infty. \tag{8}$$

In this case,

$$f(x) = \int_a^\infty g(t)x(t)\,dt$$

defines a bounded linear functional f on $R_B[a, \infty)$ such that

$$\|f\| = \int_a^\infty |g(t)|\,dt. \tag{9}$$

That $\|f\| \leqslant \int_a^\infty |g(t)|\,dt$ is trivial. To prove the reverse inequality we take any $\varepsilon > 0$. Then by (8) there exists $b = b(\varepsilon) > a$ such that

$$\int_a^\infty |g(t)|\,dt < \int_a^b |g(t)|\,dt + \varepsilon.$$

Hence (7) above implies

$$\int_a^\infty |g(t)|\,dt < (b-a)/n + \|f\| + \varepsilon,$$

and (9) follows on letting $n \to \infty$, and then letting $\varepsilon \to 0$.

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